must remain always a little behind, rather than in advance of the times, on pedagogical theory.

What we do desire is to inspire, not a discussion along these lines, but rather a discussion of definite *mathematical* problems. It is our hope that this article will spur mathematicians to the realization of the new conditions that confront us, to the variety of problems that demand discussion, to the vital importance which their solution holds to a great body of intelligent people. For the discussion from this standpoint of mathematical questions, properly speaking, the MONTHLY will open its columns without reserve, except for those usual discretionary and advisory powers that are traditional in any editorship.

The Future.—This is not the last word that the MONTHLY will have to say upon this question. We shall indulge in rather little general pedagogical preaching, for we claim no professional attitude toward that subject. What seemed necessary has been said above. For this issue this is enough. We shall, however, not place any limitation upon the scope of our suggestions to readers and to possible contributors; and we shall hope, by these suggestions, to point out very explicit problems of a mathematical nature which might well be treated. We shall aim to awaken an interest in their discussion, even among those who now regard their own profession—the teaching of collegiate mathematics—with distrust as a possible field for that type of human thinking that is known as scientific research. In this distrust, we, the editors of the MONTHLY, emphatically announce that we do not share.

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College, Colorado Springs.

I. FROM NAPIER TO LEIBNIZ AND JOHN BERNOULLI I. 1614–1712.

LOGARITHMS OF POSITIVE NUMBERS.

The seventeenth and eighteenth centuries are notable in the history of algebra for important developments of the notations and ideas which mark its modern treatment. The study of this period in algebra is more than mere pastime. In the words of Felix Klein: "If we really desire to advance to a full understanding of the theory of logarithms, it is best to follow in broad outline the history of its creation."[1]

Logarithms were invented before our modern exponential notation, a^n, was introduced into algebra. To be sure, the use of algebraic symbols that were more or less different from the modern symbols to indicate powers and roots of a number had been suggested before the advent of the logarithm, but we shall

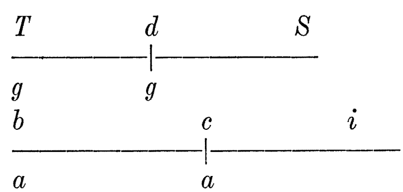
[1] F. Klein, *Elementarmathematik vom höh. Standpunkte aus*, I, Leipzig, 1908, p. 325.

see that these suggestions had remained unheeded; the fact is that the inventors of logarithms did not use the modern exponential notation and were not familiar with the exponential concept which now plays such a fundamental rôle in the development of logarithmic theory. What then were the basic considerations in the development of logarithms as entertained by their inventors, John Napier and Joost Bürgi?

John Napier's *Mirifici logarithmorum canonis descriptio*[1] appeared in 1614 in Edinburgh, and his *Mirifici logarithmorum canonis constructio* appeared there as a posthumous work in 1619, though written as early as, or earlier than, the *Descriptio*. Napier based his explanations upon two considerations: (1) The geometrico-mechanical concept of flowing points, (2) the relations which exist between arithmetic and geometric series. Several writers before the time of Napier called attention to certain relations between the terms of a geometric series and the terms of an arithmetic series, which relations involve the logarithmic idea. But those writers did not realize the possibilities of this idea nor did they conceive and execute the plan of computing a pair of corresponding series sufficiently dense for practical use in computation. From certain passages in authors like Stifel[2] one might be tempted to say that the logarithmic concept really existed before the time of Napier and Bürgi. Yet how much of a novelty the logarithms of Napier really were to the foremost mathematicians of his day can be realized by the enthusiasm with which men like Briggs and Kepler took up the new topic. So new and original did logarithms seem to Briggs that he left London for Scotland, to visit Napier, the inventor. Briggs addressed him thus:

> "My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy viz., the logarithms."

Napier lets the point g move along the definite line TS with a diminishing velocity such that its velocity at T is to that at d, as the distance TS is to the distance dS. At the same time Napier lets a point a move along the line bc (which is of indefinite length) with a uniform velocity which is the same as the initial velocity of the point g. If the two points start to move at the same moment, and if g is at d when a is at c, then the length bc is defined as the logarithm of dS. Napier constructed tables for trigonometric computation. With that end in view he lets TS stand for the *radius*, assigning to it the value 10^7, while dS stands for a given *sine*. At that time trigonometric functions were not thought of strictly as *ratios*. In the language of Napier, the definition of a logarithm is here as follows:



[1] A reprint of the entire *Descriptio* is found in Maseres' *Scriptores logarithmici*, Vol. 6, London, 1807, pp. 475–624.

[2] Tropfke, *Gesch. d. Elementar-Mathematik*, Vol. II, 1903, pp. 141–145.

"The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine."[1]

Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$\begin{array}{cccccc} v, & v(1 - 1/v), & v(1 - 1/v)^2, & \cdots & v(1 - 1/v)^v & \cdots \\ 0, & 1\quad, & 2\quad, & \cdots & v & \cdots \end{array}$$

The numbers in the upper series represent successive values of the *sines;* the numbers in the lower series stand for the corresponding logarithms. Thus, $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log (10^7\{1 - 10^{-7}\})^n = n$, where $n = 0, 1, 2, \cdots$. Napier had a definite object in view in making the logarithm of the radius, 10^7, equal to zero. He looked upon this arrangement as labor-saving, considering the frequency with which multiplications and divisions by the radius arise in trigonometric computation. He made the geometric series decrease while the corresponding arithmetic series increases. This makes the logarithms of the sines of all angles between 0° and 90° positive; in our modern tables these logarithms are negative.

In the *Descriptio* logarithms are defined as follows: *Logarithmi sunt numeri qui proportionalibus adjuncti aequales servant differentias.*" (Logarithms are numbers which correspond to proportional numbers and have equal differences.) The proportional numbers are the terms of the geometric progression; the numbers having equal differences are the terms of the arithmetic progression. The word "logarithm" is of Greek structure and signifies "number of ratios." The idea is this: $v(1 - 1/v)^n$ is gotten from v by n successive applications of the ratio $(1 - 1/v)$. Hence n, which is the logarithm, indicates "the number of ratios." Napier restates the definition in 1614 as follows: *Logarithmi dici possunt numerorum proportionalium comites æquidifferentes.* (Logarithms may be called equidifferent companions to proportional numbers.)

Not only is Napier's definition of a logarithm different from the modern definitions, but the notion of a "base" is inapplicable to his system. To force the concept of a "base" upon his system we must modify it somewhat. If each number of the two progressions of Napier is divided by 10^7, so that 0 becomes the logarithm of 1, then 1 is the logarithm of $(1 - 1/10^7)^{10^7}$, which is nearly equal to e^{-1}, where e is the base of the natural system. Hence the base of Napier's logarithms, when modified as here indicated, is very nearly e^{-1}.

It is well-known that Joost Bürgi invented logarithms independently of Napier, but he lost all rights of priority by failure to publish until the praises of Napier's book began to resound throughout Europe. In 1620 appeared in Prag the *Progress-Tabulen,* containing Bürgi's logarithmic tables, but omitting

[1] *The Construction of the wonderful Canon of Logarithms* by JOHN NAPIER. Translated from Latin into English by W. R. MACDONALD, Edinburgh and London, 1889, p. 19.

the explanations of them that were promised on the title-page.[1] Hence his logarithms were unintelligible to the ordinary reader. Common to Bürgi and Napier was the use of progressions in defining logarithms. In Bürgi's tables the numbers in the arithmetic progression were printed in *red,* the numbers in the geometric progression were in *black.* The relation between Bürgi's logarithms, $10n$, and their antilogarithms is expressed in modern notation by the equation $10n = \log [10^8(1 + 1/10^4)^n]$, $n = 1, 2, 3, \cdots$. The notion of a "base" can no more be forced upon Bürgi's logarithms than it can be upon the logarithms in Napier's tables.[2] In neither system is $\log 1 = 0$. Their logarithmic concepts were more general than those of the present day in this respect, that by sliding one progression past the other they could select any positive number at random as the one whose logarithm is zero. We have seen that Napier originally chose $\log 10^7 = 0$ while Bürgi chose $\log 10^8 = 0$. The logarithms in their tables were integral numbers. More than this, the terms of the two series could be made to increase in the same direction or in opposite directions, at pleasure. That is, if $m > n$, one can make $\log m < \log n$, or $\log m > \log n$, just as one may choose. Napier originally chose the first alternative, Bürgi the second.

It is generally known that Napier and Briggs conferred with each other and agreed to modify the original logarithms of Napier. In the *Appendix* to Napier's posthumous work, the *Constructio,* an improvement is suggested, "which adopts a cypher as the Logarithm of unity, and 10,000,000,000 as the Logarithm of either one tenth of unity or ten times unity." The subsequent use of decimal fractions in logarithmic tables led to the common logarithm proper, in which $\log 1 = 0$ and $\log 10 = 1$. A readjustment of Napier's original logarithms was made in John Speidell's *New Logarithmes,*[3] published in 1619 in London, whereby the logarithms virtually became the so-called "natural logarithms" of to-day.

Since at that time logarithms were studied in connection with some geometric progression having a positive initial term and a positive ratio between successive terms, so that negative terms could not arise in the progression, the question did not force itself for consideration, what the logarithm of a negative number should be. So far as we have noticed, no writer of the seventeenth century has raised this question. We shall see that Leibniz raised it in 1712. It would not be surprising had it arisen earlier. Charles Reyneau in his *Analyse démontrée,* Paris, 1708, Vol. II, p. 802, gives the formula for differentiation, "$dl - x = - 1dx/x$," but changes it in the table of errata to "$d - lx = - 1dx/x$." Did Reyneau fail to receive the suggestion from the printers' devil's "$l - x$"? In the seventeenth century the theory of logarithms was really on a very satisfactory

[1] These explanations were printed for the first time in *Grunert's Archiv,* Vol. 26, 1856, p. 323, in an article giving valuable historical notes relating to Bürgi.

[2] Misleading is the statement of M. Koppe: "... die Byrgschen [logarithmen] haben zur Basis $(1 + 10^{-4})^{10000}$, d. h. eine rationale Zahl, die mit e auf 4 Stellen übereinstimmt" (*Bibliotheca mathematica,* 3d S., Vol. 3, 1902, p. 151), as well as the statement of Felix Klein: "... dass die Bürgische Basis $(1{,}0001)^{10000} = 2{,}718146$ ... mit e bereits auf 3 Dezimalen übereinstimmt" (F. Klein, *Elementarmathematik von höh. Standpunkte aus,* I, Leipzig, 1908, p. 333). From Bürgi's tables we copy $\log 134983856 = 30000$. It is readily seen that the base $e = 2.718 \ldots$ does not fit here.

[3] John Speidell's book is reprinted in Maseres' *Scriptores logarithmici,* Vol. 6, 1807, pp. 728–759.

basis. The definition of logarithms was restricted so as to apply only to positive numbers; to every positive number there corresponded one and only one logarithm. The needs of the practical computer were met. No necessity arose then to extend the logarithmic concept to negative or complex numbers.

A revision of Napier's definition of the logarithm of positive numbers was made by Edmund Halley. In 1695[1] he used these words: "The Old definition of Logarithms, that they are *Numerorum proportionalium æquidifferentes comites,* is too fancy to define them fully: For they may much more properly be said to be *Numeri Rationum Exponentes.*" "Thus, if there be supposed between 1 and 10 an infinite Scale of mean Proportionals, whose Number is 100,000, etc., *in infinitum;* between 1 and 2 there shall be 30102, etc., of such Proportionals, and between 1 and 3 there will be 47712, etc., of them, which Numbers therefore are the Logarithms of the Rationes, of 1 to 10, 1 to 2, and 1 to 3; and not so properly to be called the Logarithms of 10, 2, and 3." Halley's article, from which we quote, is obscurely and inaccurately worded, but possesses the merit of giving the first derivation, by processes divorced from geometry (the hyperbola), of infinite series for the computation of logarithms.

Halley's idea of logarithms of "ratios" involves an interesting point which historians have hitherto completely overlooked. Halley anticipated Roger Cotes by nineteen years in the introduction of a certain mode of measuring a ratio. Halley considers ratio a "*quantitas sui generis.*" Yet Halley's idea finds its inception in Napier's ratios of distances of two flowing points. Halley says: "These Rationes we suppose to be measured by the Number of Ratiunculae contained in each." This number stands for the logarithm of the ratio. But "logarithms thus produced may be of as many forms as you please." If in place of 100,000 mean proportionals between 1 and 10 we take 230258 of them, then, instead of common logarithms, we get the natural logarithms. Every system of logarithms differs by a constant factor from some system chosen as a standard. Hence, for a given ratio, the number of ratiunculæ is arbitrary; it is equal to the number in the standard system, multiplied by a constant. In other words, *the measure of a ratio is a constant times the standard logarithm of that ratio.* This is the idea, here expressed by Halley, later elaborated by Cotes, then largely lost sight of until the nineteenth century, when F. Klein introduced it anew, in order that he might establish the validity of the processes of projective geometry for the hypotheses of non-euclidean geometry.[2]

The following quotation from the article "Logarithms" in the second edition (1743) of E. Stone's *New Mathematical Dictionary* is of interest, because it asserts the dependence of Cotes upon Halley and especially because it gives utterance to a feeling of strangeness toward Halley's definition of the "measure of a ratio"

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[1] *Philosophical Transactions,* London, No. 216, p. 58.

[2] Klein, *Götting. Nachrichten,* 1871, No. 17, and *Mathem. Annalen,* Vol. 4, 1871, pp. 573–625. It is possible that even before Halley this idea of the measure of a ratio may have been entertained by Kepler, for he speaks of "*proportio, ejusque mensura, logarithmus.*" See Kepler, *Tabulæ Rudolphinæ,* Ulm, 1627, Cap. III, p. 11. In fact, Charles Hutton, in his *Philosophical and Mathematical Dictionary,* London, 1815, Art. "Exponent," mentions "the idea of measures of ratios, as delivered by Kepler, Mercator, Halley, Cotes, etc."

which seizes one who has not gone beyond the elementary concepts of measurement. Stone says:

> "Mr. Cotes too, at the Beginning of his *Harmon. Mensur.* has done this Business in imitation of Dr. Halley, altho' more short, yet with the same Obscurity: for I appeal to any one, even of his greatest Admirers, if they know what he would be at in his first Problem, *viz. to find the Measure of a Ratio* from the Terms of the Problem itself. . . ."

"Logarithms the measures of ratios" is referred to by Saunderson in his *Algebra*,[1] and by W. J. G. Karsten of Halle, in an important paper on logarithms,[2] and by J. F. Lorenz, in his *Elemente der Mathematik*,[3] but in general, this definition failed to influence mathematical thought of the eighteenth century.

The early explanations of the use of logarithms involved the consideration of many special cases. The theorems $\log a + \log b = \log ab$, $\log a - \log b = \log a/b$, $\log a^m = m \log a$, were explicitly stated by Oughtred (not in algebraic symbols, as here, but in words) in his booklet *De æquationum affectarum resolutione in numeris*, which was published in 1652, bound in one volume with his *Clavis mathematica*. The 1631 edition of the *Clavis* does not give these theorems.[4]

The theoretic viewpoint of the logarithm was broadened somewhat during the seventeenth century by the graphic representation, both in rectangular and polar coördinates, of a variable number and its variable logarithm. Thus were invented the *logarithmic curve* and the *logarithmic spiral*. The graphic representation of functions was then just beginning to be understood. Probably the logarithmic curve suggested itself to many minds. It has not been possible thus far to ascertain with certainty who was the first inventor of it. Hutton says[5] that the curve has been attributed to Edmund Gunter, but Hutton could not find it in Gunter's writings. Most likely the rumor is due to a confusion between the "logarithmic curve," and the "logarithmic line" invented by Gunter and known as the forerunner of the slide rule. It has been thought that the earliest reference to the logarithmic curve was made by the Italian Evangelista Torricelli in a letter of the year 1644,[6] but Paul Tannery has made it practically certain that Descartes knew the curve in 1639.[7]

It is found in a work of J. Gregory of the year 1667[8] and is clearly explained in the second edition (1690) of a book by the French mathematician, C. F. M. Dechales. In the same year Christiaan Huygens made known without proof the beautiful properties of the logarithmic curve;[9] these properties were proved later

[1] *Select Parts of Professor Saunderson's Elements of Algebra*, 3d ed., London, 1771, p. 402.

[2] *Abh. Münch. Acad.*, V, 1768.

[3] J. F. Lorenz, *Elemente*, I Theil, Leipzig, 1793, p. 140.

[4] H. Bosmans, "La première édition de la Clavis mathematica d'Oughtred," *Annales de la Société scient. de Bruxelles*, T. 35, 2. partie, p. 38.

[5] C. Hutton, *Math. Tables*, London, 1811, Introduction, p. 84.

[6] G. Loria in *Bibliotheca mathematica*, 3d S., Vol. I, 1900, p. 75.

[7] *L'intermédiaire des mathématiciens*, T. VII, 1900, p. 95. Tannery refers to a letter of Descartes, dated Feb. 20, 1639 (Édit. of Descartes, 1677, Vol. III, letter 71).

[8] J. Gregory, *Geometriæ pars universalis*, Venetiæ, 1667; See Montucla, *Histoire des mathématiques*, II, 1799, p. 85.

[9] C. Huygens, *De la cause de la pesanteur*, published in 1690 as an appendix to the *Traité de la lumière*.

by G. Grandi[1] and G. Fontana.[2] Theorems on the quadrature of this curve were given by Torricelli, Huygens and J. Craig.[3] This curve was discussed also by J. Bernoulli.[4] Its rectification was first explained in 1692 by Marquis l'Hospital in a letter to Leibniz.[5] We shall see that during the eighteenth century the logarithmic curve plays a leading rôle in the discussion of the theory of logarithms, and that it helped to cloud rather than clarify the questions at issue.

The curve which is the graphic representation in polar coordinates of the relation between a variable and its logarithm was invented by René Descartes. It is described by him in 1638 in a letter to P. Mersenne.[6] Descartes does not give its equation, nor does he connect it with logarithms. He describes it as the curve which makes equal angles with all the radii drawn through the origin. Soon after, this spiral was re-invented by Torricelli, who, as we have seen, is second only to Descartes in being unquestionably associated with the history of the logarithmic curve. The name, *logarithmic spiral,* was coined by Pierre Varignon in a paper presented to the Paris academy in 1704 and published in 1722.[7] In England this spiral commanded the attention of Oughtred, Collins, Wallis and Barrow.[8]

No less important in the history of logarithms is a third curve, namely the hyperbola. The quadrature of the space between the hyperbola and its asymptotes was effected by Gregory St. Vincent in Book VII of his *Opus geometricum,* Antwerp, 1647.[9]

This area, for the rectangular hyperbola $xy = 1$, is expressed now in the logarithmic form $\log y_1/y$, the area thus indicated being bounded by the axis of x, the ordinates y and y_1, and the hyperbolic arc. Nevertheless, this investigation of Gregory St. Vincent, strictly speaking, does not figure in the history of logarithms. He does not mention logarithms. His result is purely geometric and would remain unaltered in every particular, had logarithms never been invented. What he established was simply the theorem that, if parallels to one asymptote are drawn between the hyperbola and the other asymptote, so that the successive areas of the mixtilinear quadrilaterals thus formed are equal, then the lengths of these parallels form a geometric progression.[10]

Apparently the first writer to state this theorem in the language of logarithms

[1] *Geometrica demonstratio theorematum Hugeniorum circa logisticam seu logarithmicam,* Florenz, 1701.

[2] *Sopra il centro di gravità della logistica finita ed infinitamente lunga,* Torino, Mem. X and XI. See also Loria, *Ebene Kurven,* Deutsche Ausgabe von F. Schütte, Leipzig, 1902, p. 543.

[3] *The quadrature of the logarithmic curve* in *Philosoph. Transactions,* No. 242, 1698.

[4] *Acta Eruditorum,* 1696, p. 216.

[5] Leibniz, *Werke,* Ed. Gerhardt, Vol. II, p. 216.

[6] *Œuvres de Descartes,* éd. Cousin, VII, Paris, 1824, pp. 336–37; Éd. Adam et Tannery, II, Paris, 1898, p. 360. See also G. Loria, *Ebene Kurven,* Leipzig, 1902, pp. 448–456.

[7] Loria, *op. cit.,* p. 444.

[8] A. Favaro, *Bibliotheca Mathematica,* N. F., 5, 1891, pp. 23–25.

[9] Karl Bopp, "Die Kegelschnitte d. Gregorius a St. Vincentio," Leipzig, 1907, in *Abhandl. z. Gesch. d. math. Wiss.* (M. Cantor), XX Heft.

[10] K. Bopp, *op. cit.,* p. 265, Propos. CXXX in book on Hyperbola. A different mode of statement is given in Propos. CXXV.

was the Belgian Jesuit Alfons Anton de Sarasa,[1] who defended Gregory St. Vincent against attacks made by Mersenne. This statement was a very natural step to take, in view of Gregory St. Vincent's theorem that, in a hyperbola $xy = 1$, the asymptotic area varies in arithmetic ratio when y or x varies in geometric ratio, and of Napier's definition of a logarithm which established a one-to-one correspondence between a geometric series and an arithmetic series.

An important publication was Nicolaus Mercator's booklet, the *Logarithmo-technia*, London, 1668. Mercator writes the equation of the hyperbola in the form $y = 1/(1 + a)$, where $1 + a$ is the abscissa, and y the ordinate. He expands $1/(1 + a)$ by division into an infinite series (in itself a novel undertaking),

$$\frac{1}{1+a} = 1 - a + aa - a^3 + \cdots.$$

He gives a crude explanation of the process of summation which we now indicate by $\int x^n dx = x^{n+1}/(n + 1)$, and then integrates the terms of the above series. One would expect him to write down the logarithmic series $\log (1 + a) = a - aa/2 + a^3/3 - \cdots$, which is attributed to him, but he does not do so. Instead of this he writes down the numerical values of the first few terms of that series, taking $a = .1$, thereby obtaining the area of the mixtilinear quadrilateral between the ordinates $y = 1$ and $y = 1 \div 1.1$ as .095310181. He repeats this process for $a = .21$.[2] Probably using the results reached by Gregory St. Vincent and de Sarasa, Mercator finally connects his own results with logarithms. That he really used the logarithmic series is evident also from the next step, which must have been obtained by integration of the terms of this series. He does not write down the general result of this integration, but again gives only the numerical values corresponding to the first few terms of the new series, for $a = .1$. Thus he obtains the value of $\int \log (1 + a)da$, for $a = .1$. These results are found at the very end of Mercator's booklet, under the caption, *Invenire summam logarithmorum*. Tropfke is of the opinion that Mercator's failure to print the general logarithmic series was due to a practice still prevalent in his day of merely hinting at new results, in order to maintain an advantage over others.[3] We do not consider this the real motive. Any reader could repeat Mercator's areal computations from the full explanations given. His course is probably due to a different conception of the proper style of exposition, which favored the concrete special case to the general formula. Wallis was the first to state Mercator's logarithmic series in general symbols.[4] While the investigations due to Gregory St. Vincent, Mercator and some others led to greatly improved methods of computing logarithms by infinite series, and to the phrasing of geometric theorems

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[1] *Solutio Problematis a R. P. Marino Mersenno propositi*, 1649. See Cantor, *op. cit.*, Vol. II, 2d ed., pp. 714, 715.

[2] See Maseres, *Scriptores logarithmici*, Vol. I, London, 1791, p. 194. Maseres reprints here the entire *Logarithmo-technia*.

[3] Tropfke, *op. cit.*, II, 1903, p. 182.

[4] Maseres, *op. cit.*, Vol. I, p. 229; *Phil. Trans.*, No. 38, year 1668.

in the language of logarithms, no modification of the logarithmic concept resulted from these researches.

In a MS. of July, 1676, Leibniz[1] derives the integral $\int dy/y$. It arose in the study of the problem which Florimond de Beaune had propounded to Descartes who replied in a letter of Feb. 20, 1639:[2] To find the quadrature of that curve in which the ordinate is to the subtangent as a given line segment is to the difference of the ordinate and abscissa. In this MS. of 1676, as well as in the published paper, the *Nova methodus*,[3] Leibniz is led by this problem to the equation $w/a = dw/dx$. He takes dx as a constant b, and gets $w = a/b \cdot dw$; that is, the ordinates w are proportional to their increments. If the x's increase in arithmetic progression, then the corresponding w's are the terms of a geometric progression. Hence the x's are the logarithms of the w's; he concluded that the curve is logarithmic.[4] As early as 1675 Leibniz used the notation "Log y," as in the equation "$a^2 - yx = 2y^2 \text{ Log } y$."[5]

The Modern Exponential Notation.

The modern symbolism for powers of numbers was introduced by René Descartes in his *La géométrie*, Paris, 1637. He writes "*aa* ou a^2 pour multiplier a par soimême; et a^3 pour le multiplier encore une fois par a, et ainsi à l'infini."[6] Thus, while Vieta represented A^3 by "*A* cubus" and Stevin x^3 by a figure 3 within a small circle, Descartes wrote a^3. In his *Géométrie* he does not use negative and fractional exponents, nor literal exponents. His notation was the outgrowth and an improvement of notations employed before him by Chuquet, Bombelli, Bürgi, Reymer, and Kepler. Chuquet's manuscript work, *Le Triparty en la science des nombres*, 1484, gives $12x^3$ and $10x^5$, and their product $120x^8$, by the symbols 12^3, 10^5, 120^8, respectively. Chuquet goes even further and writes[7] $12x^0$ and $7x^{-1}$ thus 12^0, 7^{1m}. He represents the product of $8x^3$ and $7x^{-1}$ by 56^2. Bürgi, Reymer and Kepler use Roman numerals for the exponential symbol. Bürgi writes $16x^2$ thus $\overset{\text{II}}{16}$. Thomas Harriot simply repeats the letters; he writes[8] $a^4 - 1024a^2 + 6254a$ thus: $aaaa - 1024aa + 6254a$.

Descartes's notation spread rapidly. It was used by Fr. v. Schooten in his commentary on Descartes's *Géométrie*, in the edition which appeared in Amsterdam in 1644.[9] This notation was used by Huygens and Mersenne in 1646 in their correspondence with each other,[10] and by Hudde in 1658.[11] Oughtred did

[1] C. I. Gerhardt, *Entdeckung der Differentialrechnung durch Leibniz*, Halle, 1848, pp. 53, 54.
[2] Cantor, *Geschichte der Mathematik*, Vol. II, 1892, p. 781.
[3] *Acta Eruditorum*, 1684 = Leibniz, *Werke*, Vol. V, 1858, pp. 220–226.
[4] Cantor, *op. cit.*, III, 1898; pp. 187, 188.
[5] Gerhardt, *op. cit.*, p. 35.
[6] *La géométrie de René Descartes*. Nouvelle Édition. Paris, 1886, p. 2.
[7] Chuquet, "Le triparty," *Bullettino Boncompagni*, Tomo XIII, Rome, 1880, p. 740.
[8] Harriot, *Artis analyticæ praxis*, London, 1631, p. 156.
[9] Matthiessen, *Grundzüge der Antiken u. Modernen Algebra*, Leipzig, 1878, p. 551.
[10] C. Huygens, *Œuvres*, T. I, La Haye, 1888, p. 24.
[11] Joh. Huddenii *Epist. I de reductione æquationum*, Amsterdam, 1658; Matthiessen, *op. cit.*, p. 374.

not use the modern exponents in any of his editions of his *Clavis mathematica* (London, 1631, 1648, 1652), but if Rigaud reproduced faithfully the notations in the original letters, it follows that Oughtred used positive integral exponents in his correspondence as early as 1642.[1] On Feb. 5, 1666–7, John Wallis wrote to John Collins, a proposed new edition of Oughtred's *Clavis* being under discussion: "It is true, that as in other things so in mathematics, fashions will daily alter, and that which Mr. Oughtred designed by great letters may now by others be designed by small; but a mathematician will, with the same ease and advantage, understand *Ac* and a^3 or *aaa*."[2] John Pell wrote r^2 and t^2 in a letter written in Amsterdam on Aug. 7, 1645.[3] Pascal made free use of positive integral exponents in several of his papers, particularly the *Potestatum numericarum summa,* 1654. G. Kinckhuysen used positive integral exponents in 1660, in his *Meet-Konst,* and in 1661 in his *Algebra.*[4]

[The February issue will conclude the discussion of the modern exponential notation and take up the unsuccessful attempts to create a theory of logarithms of negative numbers.]

ERRORS IN THE LITERATURE ON GROUPS OF FINITE ORDER.

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Errors in General.

Although mathematics is an exact science, the mathematical literature is disfigured by numerous errors. Some of the most influential works fail along this line. In the preface to volume 1 of his *Zahlentheorie,* 1892, Paul Bachmann observes that Gauss' *Disquisitiones arithmeticæ,* 1801, was scarcely readable on account of the large number of annoying typographical errors.

Imperfect Proofs.—While typographical errors are annoying they are not the most serious errors. A much more annoying class of errors is due to imperfect proofs. The careless use of infinite series during the seventeenth and eighteenth centuries, without any inquiry as to convergence, is one of the most important instances of imperfect methods. In fact, in the early part of the nineteenth century there were only a few mathematicians who were sufficiently careful about this matter; so that Abel could properly write, in 1826, to his friend Holmboe, that most of the papers dealing with series were inexact.[5] Recently Study called atten-

[1] S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century,* Vol. I, Oxford, 1841, p. 63.

[2] Rigaud, *op. cit.,* Vol. I, p. 475.

[3] J. O. Halliwell, *Progress of Science in England,* London, 1841, p. 89.

[4] Kinckhuysen, *De Grondt der Meet-Konst,* De Haerlem, 1660; *Algebra ofte Stel-Konst,* De Haerlem, 1661.

[5] Niels Henrik Abel Memorial, Correspondance, 1902, p. 16.